View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Le Matematiche (Dipartimento di Matematica e Informatica,...

LE MATEMATICHE
Vol. LXXI (2016) – Fasc. II, pp. 45–52
doi: 10.4418/2016.71.2.4

SUFFICIENT CONDITION FOR GENERALIZED SAKAGUCHI TYPE SPIRAL-LIKE FUNCTIONS

TRAILOKYA PANIGRAHI

In the present paper, the author defines a class of analytic generalized Sakaguchi type spiral-like functions on the open unit disk $\mathbb{U}$ and obtain certain sufficient condition for functions to be in this class. Several corollaries and consequences of the main results are also considered.

1. Introduction and Motivation

Let $\mathcal{A}_n$ denote the class of all functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}.$$

In particular, for $n = 1$ we write $\mathcal{A}_1 := \mathcal{A}$.
A function $f(z) \in \mathcal{A}_n$ is said to be starlike of order α if it satisfies the inequality

$$\Re\left[\frac{zf'(z)}{f(z)}\right] > \alpha \quad (0 \leq \alpha < 1; \;\; z \in \mathbb{U}). \tag{2}$$

Entrato in redazione: 25 maggio 2015
AMS 2010 Subject Classification: 30C45.
Keywords: Analytic function, Starlike function, Spiral-like function, Sakaguchi type function.

We denote such class by $\mathcal{S}_n^*(\alpha)$. For $n=1$, we denote such class by $\mathcal{S}^*(\alpha)$. Further, a function $f \in \mathcal{A}_n$ is said to be λ-spiral-like function of order β denoted by $\mathcal{SP}_n(\lambda,\beta)$ if and only if the following inequality holds true:

$$\Re\left[e^{i\lambda}\frac{zf'(z)}{f(z)}\right] > \beta \quad \left(0 \leq \beta < 1,\ |\lambda| < \frac{\pi}{2};\ z \in \mathbb{U}\right). \tag{3}$$

For $\beta = 0$ and $n = 1$, the class $\mathcal{SP}_1(\lambda, 0)$ reduces to $\mathcal{S}_p(\lambda)$ (see [1]). Špaěck [2] proved that members of $\mathcal{S}_p(\lambda)$ known as λ-spiral-like functions that are univalent in the unit disk $\mathbb{U}$.

Recently, Goyal et al. [3] introduced and studied the class $\mathcal{S}_n(\beta,t)$ as follows. A function $f(z) \in \mathcal{A}_n$ is said to be in the class $\mathcal{S}_n(\beta,t)$ if it satisfies

$$\Re\left[\frac{(1-t)zf'(z)}{f(z)-f(tz)}\right] > \beta \quad (|t| \leq 1,\ |t| \neq 1) \tag{4}$$

for some β $(0 \leq \beta < 1)$ and for all $z \in \mathbb{U}$.

Motivated by above mentioned work, we define the subclass of $\mathcal{A}_n$ as follows:

Definition 1.1. A function $f(z) \in \mathcal{A}_n$ is said to be in the generalized Sakaguchi type spiral-like class $\mathcal{S}_n(\lambda,\beta,s,t)$ if it satisfies

$$\Re\left[e^{i\lambda}\frac{(s-t)zf'(sz)}{f(sz)-f(tz)}\right] > \beta cos\lambda \quad (z \in \mathbb{U}), \tag{5}$$

for some β $(0 \leq \beta < 1)$, s and t are real parameters, $s > t$ and λ is real with $|\lambda| < \frac{\pi}{2}$.

By specializing the parameters $\lambda,\ n,\ s,\ t$ and β, we obtain the following subclasses studied by earlier authors. For

- $\lambda = 0,\ s = 1$, the class $\mathcal{S}_n(0,\beta,1,t) = \mathcal{S}_n(\beta,t)$ has been studied by Goyal et al. [3];
- $s = n = 1, \lambda = 0$, the class $\mathcal{S}_1(0,\beta,1,t) = \mathcal{S}(\beta,t)$ has been studied by Owa et al. [4, 5], Goyal and Goswami [6] and Cho et al. [7];
- $s = 1,\ \lambda = 0,\ n = 1,\ \beta = 0,\ t = -1$, the class $\mathcal{S}_1(0,0,1,-1) = \mathcal{S}(0,-1)$ has introduced and studied by Sakaguchi [8].

We note that for $\lambda = 0,\ \ n = 1,\ \ s = 1,\ \ t = 0$, the above class reduce to the well-known subclass of $\mathcal{A}$ consisting of univalent starlike functions of order β [9].

The object of the present paper is to obtain certain sufficient condition for a function $f \in \mathcal{A}_n$ to be in the class $\mathcal{S}_n(\lambda,\beta,s,t)$.

We need the following lemma for our investigation:

Lemma 1.2 (see [10]). *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that ϕ is a mapping from $\mathbb{C}^2 \times \mathbb{U}$ to $\mathbb{C}$ which satisfies $\phi(ix, y, z) \notin \Omega$ for $z \in \mathbb{U}$, and for all real x, y such that $y \leq \frac{-n}{2}(1+x^2)$. If the function $p(z) = 1 + c_n z^n + \cdots$ is analytic in $\mathbb{U}$ and $\phi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{U}$, then $\Re(p(z)) > 0$.*

2. Main Results

Unless otherwise stated, we assume throughout our sequel, that λ is real with $|\lambda| < \frac{\pi}{2}$, $0 \leq \beta < 1$, $n \in \mathbb{N}$, s and t are reals such that $s > t$.

Theorem 2.1. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\Re\left[\left(e^{i\lambda}\frac{(s-t)^2 zf'(sz)}{f(sz)-f(tz)}\right)\left(\frac{\alpha szf''(sz)}{f'(sz)} + \frac{\alpha tzf'(tz)}{f(sz)-f(tz)} + 1\right)\right] > \frac{Q^2}{4P} + R, \quad (6)$$

where $0 \leq \alpha \leq 1$ and

$$P = \alpha(1-\beta)\left\{\frac{n}{2}(s-t) + s(1-\beta)cos^2\lambda\right\}cos\lambda, \quad (7)$$

$$Q = \alpha s(1-\beta)(\beta cos\lambda - 1)sin2\lambda cos\lambda, \quad (8)$$

$$R = \left[\beta(1-\alpha) - \frac{n\alpha}{2}(1-\beta)\right](s-t)cos\lambda + \alpha s\beta^2 cos^3\lambda + \alpha s\left(\beta - \frac{1}{2}\right)sin\lambda sin2\lambda, \quad (9)$$

then $f(z) \in \mathcal{S}_n(\lambda, \beta, s, t)$.

Proof. Define the function $p(z)$ by

$$e^{i\lambda}\frac{(s-t)zf'(sz)}{f(sz)-f(tz)} = [(1-\beta)p(z) + \beta]cos\lambda + isin\lambda. \quad (10)$$

Then $p(z) = 1 + c_n z^n + \cdots$ is analytic in $\mathbb{U}$ with $p(0) = 1$.
Taking logarithmic differentiation on both sides of (10) with respect to z, we get after simplification

$$\frac{\alpha szf''(sz)}{f'(sz)} + \frac{\alpha tzf'(tz)}{f(sz)-f(tz)} + 1 = \frac{\alpha szf'(sz)}{f(sz)-f(tz)} + \frac{\alpha(1-\beta)zp'(z)cos\lambda}{[(1-\beta)p(z)+\beta]cos\lambda + isin\lambda} + 1 - \alpha. \quad (11)$$

Therefore, it follows that

$$e^{i\lambda}\frac{(s-t)^2zf'(sz)}{f(sz)-f(tz)}\left[\frac{\alpha szf''(sz)}{f'(sz)}+\frac{\alpha tzf'(tz)}{f(sz)-f(tz)}+1\right]$$
$$=Lzp'(z)+Mp^2(z)+Np(z)+O$$
$$=\phi(p(z),zp'(z);z)(say), \tag{12}$$

□

where

$$L=\alpha(s-t)(1-\beta)cos\lambda$$
$$M=\alpha se^{-i\lambda}(1-\beta)^2cos^2\lambda$$
$$N=(1-\beta)[(1-\alpha)(s-t)cos\lambda+\alpha se^{-i\lambda}(2\beta cos^2\lambda+isin2\lambda)]$$
$$O=(1-\alpha)(s-t)[\beta cos\lambda+isin\lambda]+\alpha se^{-i\lambda}(\beta^2cos^2\lambda-sin^2\lambda+i\beta sin2\lambda).$$

Now, for all real x and y satisfying $y\leq\frac{-n}{2}(1+x^2)$, we have

$$\phi(ix,y;z)=Ly-Mx^2+iNx+O \tag{13}$$

Taking real part on both side of (13), we have

$$\Re\phi(ix,y;z)\leq-Px^2+Qx+R$$
$$=-\left[\sqrt{P}x-\frac{Q}{2\sqrt{P}}\right]^2+\frac{Q^2}{4P}+R$$
$$\leq\frac{Q^2}{4P}+R, \tag{14}$$

where P, Q and R are given by (7), (8) and (9) respectively.
Let

$$\Omega=\{w:\Re w>\frac{Q^2}{4P}+R\}.$$

Then

$$\phi(p(z),zp'(z);z)\in\Omega \quad \text{and} \quad \phi(ix,y;z)\notin\Omega$$

for all real x and y satisfying $y\leq\frac{-n}{2}(1+x^2)$, $z\in\mathbb{U}$.
Hence by virtue of Lemma 1.2, we obtain the desired result.
If we take $\lambda=0$ in Theorem 2.1, we obtain

Corollary 2.2 (see [11]). *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\Re\left[\left(\frac{(s-t)^2 zf'(sz)}{f(sz)-f(tz)}\right)\left(\frac{\alpha szf''(sz)}{f'(sz)}+\frac{\alpha tzf'(tz)}{f(sz)-f(tz)}+1\right)\right]$$
$$> \alpha\beta\left\{s\beta+\frac{n}{2}(s-t)-(s-t)\right\}+\left\{\beta-\frac{n\alpha}{2}\right\}(s-t)$$
$$(0 \le \alpha \le 1,\ 0 \le \beta < 1,\ s > t;\ z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}_n(\beta, s, t)$.

If we take $s = 1$ in Corollary 2.2, we obtain

Corollary 2.3 (see [3]). *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\Re\left[\left(\frac{(1-t)^2 zf'(z)}{f(z)-f(tz)}\right)\left(\frac{\alpha zf''(z)}{f'(z)}+\frac{\alpha tzf'(tz)}{f(z)-f(tz)}+1\right)\right]$$
$$> \alpha\beta\left\{\beta+\frac{n}{2}(1-t)-(1-t)\right\}+\left\{\beta-\frac{n\alpha}{2}\right\}(1-t)$$
$$(0 \le \alpha \le 1,\ 0 \le \beta < 1,\ |t| \le 1,\ t \ne 1;\ z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}_n(\beta, t)$.

Taking $t = -1$ in Corollary 2.3 gives:

Corollary 2.4. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\Re\left[\left(\frac{zf'(z)}{f(z)-f(-z)}\right)\left(\frac{\alpha zf''(z)}{f'(z)}-\frac{\alpha zf'(-z)}{f(z)-f(-z)}+1\right)\right]$$
$$> \frac{\alpha\beta}{4}(\beta+n-2)+\left(\frac{2\beta-n\alpha}{4}\right)$$
$$(0 \le \alpha \le 1,\ 0 \le \beta < 1;\ z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}_n(\beta, -1)$.

By taking $\beta = 0$ in Corollary 2.4, we have

Corollary 2.5. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\Re\left[\left(\frac{zf'(z)}{f(z)-f(-z)}\right)\left(\frac{\alpha zf''(z)}{f'(z)}-\frac{\alpha zf'(-z)}{f(z)-f(-z)}+1\right)\right] > \frac{-n\alpha}{4}$$
$$(0 \le \alpha \le 1;\ z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}_n(0, -1)$.

Putting $t = 0$ in Corollary 2.3, we obtain the following result.

Corollary 2.6 (see [12]). *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\Re\left[\frac{zf'(z)}{f(z)}\left(\frac{\alpha zf''(z)}{f'(z)}+1\right)\right] > \alpha\beta\left(\beta+\frac{n}{2}-1\right)+\left(\beta-\frac{n\alpha}{2}\right)$$

$$(0 \le \alpha \le 1,\ \ 0 \le \beta \le 1;\ \ z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}_n(\beta, 0) = \mathcal{S}_n^*(\beta)$.

If we take $n = 1$ and $\beta = 0$ in Corollary 2.6, we obtain

Corollary 2.7 (see [13]). *If $f \in \mathcal{A}$ satisfies the inequality*

$$\Re\left[\frac{zf'(z)}{f(z)}\left(\frac{\alpha zf''(z)}{f'(z)}+1\right)\right] > -\frac{\alpha}{2} \quad (z \in \mathbb{U}),$$

for some α $(0 \le \alpha \le 1)$, then $f(z) \in \mathcal{S}_1(0,0) = \mathcal{S}^*$.

Taking $\lambda = 0, n = 1, \beta = \frac{\alpha}{2}$ and $s = 1$ in Theorem 2.1 yields

Corollary 2.8 (see [12]). *If $f(z) \in \mathcal{A}$ satisfies the condition*

$$\Re\left[\frac{(1-t)^2 zf'(z)}{f(z)-f(tz)}\left\{\frac{\alpha zf''(z)}{f'(z)}+\frac{\alpha tzf'(tz)}{f(z)-f(tz)}+1\right\}\right] > \frac{\alpha^2}{4}(\alpha-(1-t))$$

$$(|t| \le 1,\ \ t \ne 1,\ \ 0 \le \alpha \le 1; z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}_1(0, \frac{\alpha}{2}, 1, t)$.

Putting $t = 0$ in the Corollary 2.8. we have

Corollary 2.9. *If $f(z) \in \mathcal{A}$ satisfies the condition*

$$\Re\left[\frac{zf'(z)}{f(z)}\left(\frac{\alpha zf''(z)}{f'(z)}+1\right)\right] > -\frac{\alpha^2}{4}(1-\alpha) \quad (z \in \mathbb{U}),$$

for some α $(\alpha \ge 0)$, then $f(z) \in \mathcal{S}_1(0, \frac{\alpha}{2}, 1, 0) = \mathcal{S}^*(\frac{\alpha}{2})$.

Theorem 2.10. *If $f(z) \in \mathcal{A}_n$ satisfies the condition*

$$\Re\left[e^{i\lambda}\frac{f(z)}{z}\left(\frac{\alpha zf'(z)}{f(z)}-\alpha+1\right)\right] > \frac{-n\alpha}{2}(1-\beta)cos\lambda+\beta cos\lambda, \tag{15}$$

then

$$\Re\left[e^{i\lambda}\frac{f(z)}{z}\right] > \beta cos\lambda \tag{16}$$

Proof. Consider

$$e^{i\lambda}\frac{f(z)}{z}=[(1-\beta)p(z)+\beta]cos\lambda+i\sin\lambda. \tag{17}$$

Taking logarithmic differentiation on both sides of (17) with respect to z and after simplification, we get

$$\begin{aligned}&e^{i\lambda}\frac{f(z)}{z}\left(\frac{\alpha zf'(z)}{f(z)}-\alpha+1\right)=\alpha(1-\beta)cos\lambda\ zp'(z)\\&+[(1-\beta)p(z)+\beta]cos\lambda+i\sin\lambda=\phi(p(z),zp'(z);z).\end{aligned} \tag{18}$$

Therefore, for all real x and y satisfying $y\leq\frac{-n}{2}(1+x^2)$, we obtain

$$\phi(ix,y;z)=\alpha(1-\beta)ycos\lambda+[(1-\beta)ix+\beta]cos\lambda+isin\lambda. \tag{19}$$

Taking real part on both sides of (19), we have

$$\begin{aligned}\Re\phi(ix,y;z)&=\alpha(1-\beta)ycos\lambda+\beta cos\lambda\\&\leq\alpha(1-\beta)cos\lambda\left(-\frac{n}{2}(1+x^2)\right)+\beta cos\lambda\\&=-\frac{n\alpha}{2}(1-\beta)x^2cos\lambda-\frac{n\alpha}{2}(1-\beta)cos\lambda+\beta cos\lambda\\&\leq\frac{-n\alpha}{2}(1-\beta)cos\lambda+\beta cos\lambda.\end{aligned} \tag{20}$$

Let $\Omega=\left\{w:\Re w>-\frac{n\alpha}{2}(1-\beta)cos\lambda+\beta cos\lambda\right\}$. □

Then from (15), (18) and (20) we obtain $\phi(p(z),zp'(z);z)\in\Omega$ and $\phi(ix,y;z)\notin\Omega$ for all real x and y satisfying $y\leq-\frac{n}{2}(1+x^2)$. Hence by application of Lemma 1.2, we obtain the desired result. The proof of Theorem 2.10 is thus completed.

Acknowledgements

The author thanks the referees for their careful reading, valuable suggestions and comments, which helped to improve the presentation of this paper.

REFERENCES

[1] T. Mathur - R. Mathur - D. Sinha, *Characterization and subordination properties for λ-spirallike generalized Sakaguchi type functions*, Palestine J. Math. 3 (1) (2014), 70–76.

[2] L. Špaěck, *Contribution à la théorie des functions univalents (In Crenz)*, Časop. Pest. Mat. Fys. Math. Math. Sci. 62 (1932), 12–19.

[3] S. P. Goyal - P. Vijaywargiya - P. Goswami, *Sufficient conditions for Sakaguchi type functions of order β*, European J. Pure Appl. Math. 4 (3) (2011), 230–236.

[4] S. Owa - T. Sekine - R. Yamakawa, *On Sakaguchi type functions*, Appl. Math. Comput. 187 (1) (2007), 356–361.

[5] S. Owa - T. Sekine - R. Yamakawa, *Notes on Sakaguchi functions*, RIMS Kokyuroku 1414 (2005), 76–82.

[6] S. P. Goyal - P. Goswami, *Certain coefficient inequalities for Sakaguchi type functions and applications to fractional derivative operator*, Acta Univ. Apulensis Math. Inform. 19 (2009), 159–166.

[7] N. E. Cho - O. S. Kwon - S. Owa, *Certain subclasses of Sakaguchi functions*, Southeast Asian Bull. Math. 17 (1993), 121–126.

[8] K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan 11 (1959), 72–75.

[9] P. L. Duren, *Univalent Functions*, Graduate Texts in Mathematics 259, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[10] S. S. Miller - P. T. Mocanu, *Differential subordinations and inequalities in the complex plane*, J. Differ. Equations 67 (1987), 199–211.

[11] T. Mathur - R. Mathur - D. Sinha, *Sufficient conditions for generalized Sakaguchi type functions of order β*, TWMS J. App. Eng. Math. 4 (1) (2014), 127–131.

[12] V. Ravichandran - C. Selvaraj - R. Rajalakshmi, *Sufficient conditions for starlike functions of order α*, J. Inequal. Pure Appl. Math. 3 (5) (2002), 1–6.

[13] J.-L. Li - S. Owa, *Sufficient conditions for starlikeness*, Indian J. Pure Appl. Math. 33 (2002), 313–318.

TRAILOKYA PANIGRAHI
Department of Mathematics
School of Applied Sciences,
KIIT University,
Bhubaneswar-751024,
Orissa, India
e-mail: `trailokyap6@gmail.com`